

last updated 2011 November 9

Reader comments
and
additional notes
on the book

The Language of Mathematics: Utilizing Math in Practice

by

Robert Laurence Baber

published by

John Wiley & Co., Inc., Hoboken, New Jersey

September 2011

Note: Wherever “the book”, “this book”, etc. appears
in the comments and additional notes below,
“book” refers to and means the book

The Language of Mathematics: Utilizing Math in Practice.

1. The logical equivalence function and its infix symbol \equiv

In Boolean expressions the infix function symbol \equiv is sometimes used instead of $=$ or \Leftrightarrow . Applied to the domain $\mathbb{B} \times \mathbb{B}$ these three functions all have the same values. Their binding orders differ: \equiv is normally defined to be either in binding order 9 (together with \Rightarrow , \Leftarrow , and \Leftrightarrow) in TABLE 3.4.2-1 on page 45 of the book or in a new binding order 10. Writing \equiv instead of $=$ reduces the need for parentheses in many Boolean expressions. Writing \equiv instead of \Leftrightarrow better expresses visually the fact that an equivalence relation is meant.

2. Terminology for derivatives

One often refers to “the derivative of $f(x)$ with respect to x ” as in the first sentence at the top of page 113 in Section 4.5 of the book. This type of phrase can be seen in several forms in the literature, for example:

1. the derivative of $f(x)$ with respect to x
2. the derivative of the function $f(x)$ with respect to x
3. the derivative of the expression $f(x)$ with respect to x
4. the derivative of an expression with respect to x
5. the derivative of the function f with respect to x
6. the derivative of the function f with respect to its argument
7. the derivative of the function f with respect to its first argument
8. the derivative of the function f with respect to its second argument
9. the derivative of the function f with respect to its ... argument

These phrases occur in the interpretation in English of mathematical expressions and parts of mathematical models. As such, they are subject to the “normal” ambiguities of English.

Strictly speaking, $f(x)$ is not a function, it is an expression referring to the value of the function f for an argument value that is the value of the variable x . Therefore, phrase 3 above is generally preferred over phrase 2. The form of phrase 2 emphasizes that the function f is of primary interest and includes the information that x is the name of the variable that is both the argument of the function and the variable with respect to which the derivative is being taken. This is a good example of hidden information, ambiguity, and emphasis, all of which are common in natural language text.

In phrase 1 above, the part “of $f(x)$ ” implies “of expression $f(x)$ ”, so phrase 1 can be viewed as an abbreviated form of phrase 3.

Phrase 5 above is reasonable only if it is clear from the context that x is the argument of the function f . If this is not clear from the context, then this phrase is ambiguous and unacceptable.

When the derivative is a partial derivative, the word “partial” should be included in the phrases above.

In summary, the preferred notational descriptions for a derivative are 3, 4, and 7 through 9. If the function f has only one argument, then phrase 6 is also among the preferred descriptions. Phrase 1, an abbreviated form of phrase 3, is also unambiguous and fully acceptable.

3. Comments from Prof. Dr.-Ing. Adolf J. Schwab
received 2011 September 29

Dear Bob,

thank you very much for the complimentary copy of your book "Language of Mathematics". This is a monument! I do agree that your book can inspire many teachers of mathematics in conveying a better understanding and appreciation.

Unfortunately, I am afraid that you may face some gusty winds from puristic math professors regarding mathematical rigor. For example, in Chapter 4.5 Calculus you do not distinguish between incremental quantities Δx and infinitesimal quantities dx . You state that "Each letter d indicates the "difference" or "differential" of the following value....." However, there is a distinct difference between Δx and dx . You can see Δx and represent it in a graph as a very small but finite length, but you cannot see dx . Δx becomes dx for Δx striving towards zero! dx is the result of taking a limit! Therefore, your equation 4.5-4, showing $dx \rightarrow 0$ according to usual nomenclature may provoke criticism (see correct writing handwritten at the end of this text). For mathematical rigor, you always have to start with an incremental, finite Δx , then take the limit for Δx going to zero. Although dx is infinitely small, a ratio of d "operand"/ dx remains finite.

Further, an indefinite integral must in its general definition always include a Constant C . This constant C is essential and cannot only show up in the accompanying text.

Finally, in your Chapter on Probability Theory I am missing the concept of the density function etc.

Whatsoever, I consider your book a highly valuable contribution and challenge to improving the didactics of mathematics. It provides a lot of food for thought of how teaching math could be done better, congratulations!!!!!!!!!!!!!!

Hoping that you are doing well I remain with warm regards

Adolf

Prof. Dr.-Ing. Adolf J. Schwab
Excellent Retired Scientist
University of Karlsruhe
Institute of Electric Energy Systems and High-Voltage Technology
Kaiserstrasse 12
D-76128 Karlsruhe
Germany

e-mail: a.schwab@ieee.org
e-mail: schwab@kit.edu
www: <http://www-ieh.etec.uni-karlsruhe.de/>
Tel: +49 721 608-2520
Fax: +49 721 695224

Equ 4.5-4 must read

$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

In taking the limit, the "difference ratio" becomes the "differential" ratio, i.e. the slope in x .

3a. The author's response to Prof. Dr.-Ing. Adolf J. Schwab's comments

I thank Prof. Dr.-Ing. Adolf J. Schwab very much for his encouraging statements about this book.

Second paragraph in Prof. Schwab's letter

The text in Prof. Schwab's paragraph on dx and Δx is very similar to the material from which I learned derivatives many years ago, so I agree with these comments. This paragraph raises several didactically interesting and useful points and issues in nomenclature, mathematical terminology in English, and mathematical notation. These points and issues deserve elaboration. Prof. Schwab's second paragraph raises some of them explicitly and others, indirectly and implicitly.

In preparation for my responses to this paragraph, I refer the reader to Section 6.10 of the book, pages 193 to 195. There it is pointed out that mathematical text is typically written in a mixture of (1) purely mathematical expressions, (2) commentary in English, and (3) the interpretation in English of mathematical expressions and parts thereof. Section 6.10 further points out that the reader should distinguish clearly between these three components of mathematical text. I will refer to these three categories of components of mathematical texts in some of my responses to the particular points and issues below.

defining a derivative: In Section 4.5, on page 113 of the book, the right side of equation [4.5-4] gives one common form for the definition of the derivative of the expression $f(x)$ with respect to the variable x . Another common form for the definition of this derivative follows from equation [4.5-1]:

$$\frac{df(x)}{dx} = \lim_{x_1 \rightarrow x} (f(x_1) - f(x)) / (x_1 - x)$$

Both forms are found in the mathematical literature, as are variations of them, for example, the above form with x instead of x_1 and x_0 instead of x :

$$\frac{df(x_0)}{dx_0} = \lim_{x \rightarrow x_0} (f(x) - f(x_0)) / (x - x_0)$$

These three forms (equation [4.5-4] in the book and the two equations above) are all equivalent ways of defining a derivative. In the case of the last equation above, x_0 is the free variable, instead of x in the other two, so the derivative is a function of the function f and the value of the variable x_0 instead of x .

Notice that neither the name " dx " nor the name " Δx " appears in either of the two definitions above.

The definitions themselves are to the right of the $=$ symbol in each expression above. The term to the left of the $=$ symbol in each expression is a notational form for the derivative of an expression with respect to a variable. That notational form should be construed as a composition of a derivative symbol and the arguments of the derivative function (see the paragraphs on **notational forms for derivatives** below).

Notice also that the variable dx in equation [4.5-4] in the book, Δx in Prof. Schwab's comments, x_1 in the first equation above, and x in the second equation above are quantified, bound, "dummy", "running" variables. This fact can probably be seen most clearly in equation [4.5-5] in the book, in which dx is quantified in the innermost quantified expression. As pointed out in Section 3.4.8 at the bottom of page 67 of the book, the name of such a

variable can be changed without affecting the value of the expression in which it appears, provided that the new name is not the same as any other name appearing in the expression.

Therefore, the variable names dx in the book, Δx in Prof. Schwab's comments, x_1 above, and x in the second equation above can be changed to any other name not already appearing in the expression in question. Whether or not the letter "d" or "Δ" appears in the name of any of these variables is irrelevant and of no consequence; mathematically, those variables could just as well have been named p , y , z , abc , or whatever. The names dx and Δx are suggestive of meanings useful to human readers, but those names are of no mathematical significance. Only the properties of the values of the variables are of mathematical significance.

It is useful to examine equation [4.5-5] in the book and identify the names of the free (unbound) variables and functions. In addition to the set \mathbb{R} , which can be viewed as a constant as is the value 0, and the standard mathematical functions \wedge , \vee , \in , $>$, $<$, $|\dots|$, $-$ and $/$, the names of the free variables and functions are f , x , and g . Therefore, if $g(x)$ exists, (that is, if any $g(x)$ satisfies equation [4.5-5]), $g(x)$ can depend only on f and x . This is a general result: the derivative of an expression with respect to a variable depends only on the expression and (the value of) that variable. This has implications for notational forms for derivatives.

notational forms for derivatives: Some of the many notational forms for derivatives appearing in the literature are listed below. As stated in the book in Section 4.5 in expression [4.5-3] and in the text at the bottom of page 112, the most commonly used notational forms are the first two listed below:

1. $\frac{df(x)}{dx}$
2. $df(x)/dx$
3. $f'(x)$
4. $\dot{y}(x)$
5. $D_x f$
6. $D_x f(x)$
7. df/dx
8. $\frac{d}{dx}f(x)$

The most common forms 1 and 2 as well as forms 7 and 8 above have a disadvantage: the sequence of letters "dx" is visually present, suggesting the variable name dx and misleading some into thinking that some residual of this variable still has an effect on the result. An advantage of these forms is that they reflect and remind the reader of the form of the definition in equation [4.5-4] on page 113 of the book. However, these forms would perhaps be better read as

$$\frac{d}{d}(f(x), x)$$

or $(d/d)(f(x), x)$ to remind the reader that the derivative of an expression $f(x)$ with respect to the variable x is a function of the expression $f(x)$ and the value of the variable x , and that the variable dx plays no role in the result itself (only in the way that result is determined). The author has never seen this notational form in the mathematical literature.

In any case, the reader is well advised to view the "d" in the denominator of forms 1, 2, 7, and 8 as a part of a derivative notation "d/d" and not as part of a variable name dx . The "x" in the denominators of these forms should be viewed as the variable name x itself, completely divorced from the "d".

Form 6 above probably expresses most clearly the fact that the derivative of an expression $f(x)$ with respect to x is a function of that expression and the variable named x . Despite this advantage, form 6 is not in widespread use. The similar notational form $D(f(x), x)$ would have the same advantage, but the author has never seen this form in the literature.

dx vs. Δx as a variable name: In Section 4.5 of the book, dx is used as the name of the variable that is named Δx in Prof. Schwab's letter. As concluded in the paragraphs on **defining a derivative** beginning on page 4 above, the name of this variable, which is a quantified variable in the expressions in which it appears, is irrelevant and of no consequence outside the expression defining the derivative in question.

Prof. Schwab distinguishes between the meanings of the names dx and Δx . The text in the book does not make this distinction. The concept of dx as described in his letter is absent from the book, see the paragraphs on "incremental" and "infinitesimal" below. In summary, if readers find this distinction useful and helpful in learning this material, fine. Mathematically it is not necessary, however, to make or to consider this distinction, and readers who find it confusing are best advised to disregard it. The notion of a variable or an expression approaching zero as a limit is sufficient (and necessary).

incremental vs. infinitesimal, difference vs. differential, infinitely small: These terms appear not only in Prof. Schwab's letter, but also in various texts on the subjects of calculus, derivatives and integration.

I will consider individually these terms and their definitions given in three of my favorite dictionaries:

- *Mathematics Dictionary, Multilingual Edition* by James and James, D. Van Nostrand Company, Inc., 1968
- *Webster's New World College Dictionary*, Third Edition, Victoria Neufeldt, Editor in Chief and David B. Guralnik, Editor in Chief Emeritus, Macmillan, 1995
- *Webster's New Collegiate Dictionary*, G. & C. Merriam Co., 1949

Where a mathematically oriented definition is given in a general dictionary, I will concentrate on that definition and pay less or no attention to other definitions clearly intended for a broader use.

As a general warning in advance, one should read critically the definitions of terms used in a mathematical context when such definitions are given in dictionaries aimed at a general readership.

incremental, difference: The terms "incremental" and "difference" are straightforward and their meanings are comparatively clear and unproblematic. For "incremental", dictionary definitions mention a difference between two numbers or increases and decreases in the value of some variable. For "incremental" some dictionaries include the statement that the increase or decrease is "usually small". None of these definitions mention a limit.

infinitesimal: In the mathematical sense, all three dictionaries mentioned above give the definition "a variable which approaches zero as a limit" or some minor rewording thereof. The variable dx in the book and the variable Δx in Prof. Schwab's letter both fulfill this definition, so they can be called infinitesimals. The two general dictionaries listed above also give such definitions as "too small to be measured", "infinitely small", "immeasurably or incalculably small", "very minute". "Too small to be measured" is clear, but is of consequence only in practical, but not in theoretical work. It is not at all clear to me what "incalculably small" is intended to mean; one can calculate with arbitrarily small numbers. The term "very minute" is very vague in meaning; while it has its place in normal natural language, it does not belong in a mathematically precise text.

Notice that the primary difference between the definition of "incremental" and "infinitesimal" is that an infinitesimal approaches zero as a limit, while the term "incremental" does not imply or necessarily involve any variable approaching any limit.

infinitely small: Linguistically, the phrase “infinitely small” is odd. I found no definition of this phrase in my dictionaries, so examined “infinitely” in order to associate its definitions with the relatively clear adjective “small”. Among the definitions of “infinitely” or its adjective form “infinite” are: “becoming large beyond any fixed bound”, “indefinitely large”, “very great”, “vast”, “immense”, “immeasurable”, “indefinitely large”, etc. With the exception of one interpretation of “immeasurable”, all of these definitions clearly indicate something large or great, which clearly contradicts the idea of “small”. Therefore, the phrase “infinitely small” is, in itself, self-contradictory and therefore meaningless, particularly in a mathematical context.

The term “immeasurable” could mean either immeasurably large or immeasurably small. The meaning “immeasurably large” belongs to the group already discussed in the paragraph above. The meaning “immeasurably small” is the same as “too small to be measured” or “incalculably small”, both of which were discussed in the first paragraph on **infinitesimal** above.

Probably “infinitely small” is intended to mean “very small” and the word “infinitely” is used instead of “very” or “extremely”.

Alternatively, “infinitely small” could mean “infinitesimally small” or, more simply, “infinitesimal”. Then a variable approaching zero as a limit is meant, see the paragraphs on **infinitesimal** above.

differential: Probably the best definition of the word “differential” in the context of calculus is “the derivative of a function multiplied by a small increment of the independent variable”. Notice the word “small”. In the common mathematical use of this word in calculus it applies to an expression in which the small increment is understood to approach zero as a limit. Typical examples are the subexpression

$$f(x) dx$$

in expression [4.5-7] on page 114 and other similar subexpressions on following pages of Section 4.5 of the book.

In an alternative definition of the integral [4.5-7] on page 114 of the book the integral is defined or interpreted to be the area under the graph of $f(x)$ (cf. the graph at the bottom of page 115 of the book). This area is approximated by rectangles of height $f(x)$ and width dx , i.e. each with area $f(x)dx$. As dx approaches zero as a limit the sum of these terms (areas of the rectangles) approaches the area under the graph as a limit.

If $f(x)$ is bounded, then every $f(x)dx$ also approaches zero as a limit as dx approaches zero as a limit, so both dx and $f(x)dx$ satisfy the definitions of the terms “differential” and “infinitesimal”.

“You can see Δx and represent it in a graph as a very small but finite length, but you cannot see dx ”: I would express this differently: One can (at least in principle) see a non-zero length on a graph, but one cannot see the length of the limit, which is zero. Also, instead of the word “finite”, the word “non-zero” would be better. See also my discussion of **dx vs. Δx as a variable name** on page 6 above.

“Each letter “d” indicates the “difference” or “differential” of the following value”: This quotation from the book (Section 4.5, last sentence on page 112) falls in category (2) on page 4 above. This sentence is intended to motivate the choice of the name of the variable dx for the difference between two values of x , and to motivate the reference $df(x)$ to the difference between the values of the expression $f(x)$ for two different values of x . Notice that the phrase “ $df(x)$ ” is not itself a mathematical expression. If it were, it would be the value of a function df for the argument value x .

Notice that both of the differences mentioned in the paragraph above are in a context in which they approach zero as a limit, so they also satisfy the definitions of an infinitesimal and of a differential.

See the comments on **notational forms for derivatives** on page 5 above, especially the suggestion to view the notation $df(x)/dx$ as

$$\frac{d}{d}(f(x), x)$$

separating distinctly the two occurrences of “d” from “f(x)” and “x”.

Third paragraph in Prof. Schwab’s letter

the constant of integration: Prof. Schwab’s comment is correct. In Section 4.5, on pages 114 and 115 of the book, I made two points regarding two functions f and g :

1. If f is the derivative of g , then g is the indefinite integral of f , and vice versa.
2. Under these conditions, f is uniquely determined by g , but g is not uniquely determined by f . Any function g_1 that differs from g only by a constant is also an indefinite integral of f .

For reasons of clarity, especially for readers encountering this topic for the first time, I chose to complete making point 1 before starting on point 2. It would have been more precise to replace “ g is the integral of f ” by “ g is an integral of f ”, but this would have implicitly introduced point 2 before point 1 was complete.

In any case, equation [4.5-8] on page 114 of the book is correct in that it gives one possible $g(x)$ for a given $f(x)$. It is not, however, the only function of x that satisfies equation [4.5-8]. If $g(x)$ satisfies equation [4.5-8], then so does $g(x)+c$, where c is any constant, as is pointed out in the first paragraph on page 115 of the book. In fact, c may be any expression not dependent on the value of the variable x .

Fourth paragraph in Prof. Schwab’s letter

probability density function: In Section 4.6.1 of the book, at the end of the last complete paragraph on page 119, is the sentence “Otherwise, and particularly if the sample space S is infinite and not countable, other approaches to defining the set A and the probability function p must be employed”. In the case of a sample space involving one random variable with values in \mathbb{R} , a probability *distribution* function P_{dist} is defined such that $P_{\text{dist}}(x)$ is the probability that the value of the random variable is less than or equal to x . The probability *density* function is the derivative of $P_{\text{dist}}(x)$ with respect to x . The set A of subsets of S is defined in an appropriate way (often as Borel sets). If the random variable takes on values in some set other than \mathbb{R} , then the set A and the probability distribution and density functions are typically defined in a corresponding way. These aspects of probability theory go beyond the scope of this book, as do many aspects of statistics, so these topics were not included. The interested reader is referred to the many books on probability theory and statistics.

In closing

I thank Prof. Dr.-Ing. Adolf J. Schwab very much for his encouraging comments and for his detailed critique of certain points in my book. I hope that my elaboration on points in his critique will help readers in their efforts to understand thoroughly some finer points of the mathematics involved and especially the corresponding English terminology.

(end of the author’s response to Prof. Dr.-Ing. Adolf J. Schwab’s comments)